by G and its determinant by $|G|$. We assume that the degenerate system ($\epsilon \doteq 0$) has a solution which can be continued up to $t = \tau$ where τ is such that for the solution in question $|G(x(\tau), u(\tau), \tau, 0)| = 0$. Part of our sufficient condition for the perturbed system to have a solution is the requirement that the characteristic equation

$$|g_{ij}(x(\tau), u(\tau), \tau, 0) - \lambda\delta_{ij}| = 0$$

have $\lambda = 0$ as a simple root.

[1] M. Nagumo, Über das Verhalten der Integrals von $\lambda y'' + f(x, y, y', \lambda) = 0$ für $\lambda \to 0$. *Proc. Phys. Math. Soc. Japan*, **21**, 529–534 (1939). I. M. Volk, A Generalization of the Method of Small Parameter in the Theory of Non-Linear Oscillations of Non-Autonomous Systems. *C. R. (Doklady) Acad. Sci. U.S.S.R.*, **51**, 437–440 (1946). Volk considers (1) where the X_i are meromorphic functions of ϵ for small ϵ and periodic in t. K. O. Friedrichs and W. R. Wasow, Singular Perturbations of Non-Linear Oscillations. *Duke Math. Jour.*, **13**, 367–381 (1946). Here the X_i are not functions of t and for $i \leqq n - 1$ are not functions of ϵ. X_n contains ϵ in the form of a factor $1/\epsilon$.

[2] D. A. Flanders and J. J. Stoker, *The Limit Case of Relaxation Oscillations, Studies in the Linear Vibration Theory*, New York Univ., 1946.

[3] This is the system, except that t is not necessarily excluded from the right members, which is considered by Friedrichs and Wasow, *loc. cit.*

[4] See Friedrichs and Wasow, and Volk, *loc. cit*, for continuous cases where right members do not and do, respectively, depend on t.

A MINIMUM PROBLEM ABOUT THE MOTION OF A SOLID THROUGH A FLUID

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1. An incompressible frictionless fluid of uniform density ρ fills the whole space outside a moving solid and is at rest at infinite distance. The motion of the solid is one of pure translation. The magnitude of the velocity is U, its direction cosines with respect to a coördinate system fixed in the solid λ, μ, ν. The kinetic energy of the fluid is of the form

$$T = {}^1/_2 MU^2.$$

The quantity M, called the virtual mass, depends on the direction of the velocity:

$$M/\rho = A\lambda^2 + B\mu^2 + C\nu^2 + 2A'\mu\nu + 2B'\nu\lambda + 2C'\lambda\mu.$$

A, B, C, A', B', C' are uniquely determined if the shape and size of the solid and the relative location of the coördinate system and the solid are given.

A closer study of the dependence of A, B, C, A', B' and C' on geometric data may seem desirable.[1] Taking a first step in such a study, we consider the *average virtual mass* $\bar{M}$, obtained by averaging M over all directions λ, μ, ν and assuming $\rho = 1$:

$$\bar{M} = (A + B + C)/3.$$

$\bar{M}$ is independent of the location of the coördinate system and depends only on the size and shape of the solid. It is easy to show that *of all ellipsoids with given volume the sphere has the minimum average virtual mass.* It would be natural to suspect that this statement remains true if for "ellipsoids" we substitute "solids." At any rate, I shall prove the analogous general theorem in two dimensions.

2. We consider now the two-dimensional motion of an incompressible frictionless fluid of uniform density ρ that fills the space around a cylinder of infinite length. The motion is parallel to a plane, the plane of the complex variable z, that is perpendicular to the cylinder and intersects it in a closed curve C (the notation of section 1 has been dropped). The exterior of C is mapped conformally onto the exterior of the unit circle in the ζ-plane so that the points at infinity correspond. Thus, z moving outside C is represented by the series

$$z = \lambda\left(\zeta + c_0 + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \ldots\ldots\right) \tag{1}$$

convergent for $|\zeta| > 1$. The number λ is positive.

We begin with the case in which the motion of the fluid at infinite distance is parallel to the real axis and has the velocity U (uniform flow disturbed by a fixed cylindrical obstacle). The corresponding motion in the ζ-plane, around a circular cylinder and with velocity Uλ at infinity, has the complex potential

$$\chi' = U\lambda\left(\zeta + \frac{1}{\zeta}\right). \tag{2}$$

Yet (2) represents also the complex potential for the z-plane provided that z and ζ are linked by the mapping (1) that transforms streamlines into streamlines and, especially, the unit circle of the ζ-plane into C.

3. To the motion just considered we add a uniform velocity U directed along the *negative* real axis. We obtain thus a new motion (disturbance of a fluid which is at rest at infinite distance by a cylinder moving through it sidewise to the left). The complex potential of this motion is obviously

$$\chi = \chi' - Uz = U\left[\lambda\left(\zeta + \frac{1}{\zeta}\right) - z\right] \tag{3}$$

where z and ζ remain linked by (1). (Of course the coördinate system remains fixed with respect to the solid.) The velocity at the point z is $\bar{w}$, conjugate to

$$w = \frac{d\chi}{dz} = U\left[\lambda\left(1 - \frac{1}{\zeta^2}\right)\frac{d\zeta}{dz} - 1\right]. \tag{4}$$

The kinetic energy of a layer of the fluid, of unit thickness and parallel to the $z = x + iy$ plane, is

$$\frac{1}{2}\rho\int\int|w|^2\, dx\, dy = \frac{1}{2}\, MU^2. \tag{5}$$

The integral is extended over the exterior of C, and M is the *virtual mass per unit height.* From (4) and (5) we obtain

$$\begin{aligned} M/\rho &= \int\int\left|\lambda\left(1 - \frac{1}{\zeta^2}\right)\frac{d\zeta}{dz} - 1\right|^2 dx\, dy \\ &= \int\int\left|\frac{dz}{d\zeta} - \lambda\left(1 - \frac{1}{\zeta^2}\right)\right|^2 d\xi\, d\eta; \end{aligned} \tag{6}$$

the latter integral is extended over the exterior of the unit circle in the $\zeta = \xi + i\eta$ plane. Introducing (1) and polar coördinates, we obtain from (6) in the usual way that

$$M/\rho = \pi\lambda^2(|c_1 - 1|^2 + 2|c_2|^2 + 3|c_3|^2 + \ldots). \tag{7}$$

Now the area of C or, what is numerically the same, the volume V of the moving cylinder per unit height is

$$V = \pi\lambda^2(1 - |c_1|^2 - 2|c_2|^2 - 3|c_3|^2 - \ldots). \tag{8}$$

This is well known and obtained by a computation analogous to the one just sketched. It follows from (7) and (8) that

$$V + M/\rho = 2\pi\lambda^2(1 - \Re c_1). \tag{9}$$

where $\Re c_1$ denotes the real part of c_1.

4. Now, we wish to obtain M_α, the virtual mass per unit height corresponding to a direction of the velocity that includes the angle α with the direction just considered. We reduce this problem to the foregoing by a rotation, introducing the new complex variables z' and ζ',

$$z' = e^{i\alpha}z, \qquad \zeta' = e^{i\alpha}\zeta.$$

We obtain from (1) that

$$z' = \lambda\left(\zeta' + c_0e^{i\alpha} + \frac{c_1e^{2i\alpha}}{\zeta'} + \frac{c_2e^{3i\alpha}}{\zeta'^{\,2}} + \ldots\ldots\right) \tag{1'}$$

Substituting $c_1e^{2i\alpha}$ for c_1 in (9), we obtain

$$V + M_\alpha/\rho = 2\pi\lambda^2(1 - \Re c_1 e^{2i\alpha}) \qquad (9')$$

and hence

$$V + M_{\alpha+\pi/2}/\rho = 2\pi\lambda^2(1 + \Re c_1 e^{2i\alpha}) \qquad (10)$$

We define $\bar{M}$, *the average virtual mass per unit height* by

$$\bar{M} = (1/2\pi\rho)\int_0^{2\pi} M_\alpha d\alpha = (1/2\rho)(M_\alpha + M_{\alpha+\pi/2}). \qquad (11)$$

($\bar{M}$ has, in fact, the dimension of an area, and so has V.) From (9′), (10) and (11) we find finally

$$V + \bar{M} = 2\pi\lambda^2 \qquad (12)$$

5. Now λ is the so-called outer radius of C (that is the radius of the circle onto the exterior of which the exterior of C is so mapped that the points at infinity correspond to each other with unit magnification). It follows from (8) (and is well known) that

$$V < \pi\lambda^2.$$

unless C is a circle. Therefore, by (12),

$$\bar{M} > V$$

with the same proviso. For the circle, however, $\bar{M} = V$. Thus, we have proved *that of all cylinders having the same area of the cross-section, the circular cylinder has the minimum average virtual mass per unit height.*

We can derive another result from (12): *the average virtual mass per unit height decreases by symmetrization.* Indeed, we know that symmetrization leaves V unchanged and decreases the outer radius λ.[2]

[1] This is suggested by a systematic study of the dependence of the capacity on geometric data which has been undertaken recently by Mr. G. Szegö and the author.

[2] See G. Pólya and G. Szegö, "Inequalities for the Capacity of a Condenser," *Amer. Jour. Math.*, **67**, 1–32 (1945), especially pp. 13–14.